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Recovery of the Symbol of Densely Defined Toeplitz Operators over the Hardy Space

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Abstract

While the symbolic map for the Toeplitz collection is well studied, little work was done on a high-defined symbolic map. Operators of Toeplitz. A family of applicant symbols includes this work. To play the Toeplitz operator symbol densely defined. Sarason sub-symbols will be introduced. This results in a partial response. In 2008, Donald Sarason asked a question. In the limited case, an operator's Toeplitzness can be categorized according to his Sarason Sub-Symbols. This explains why the Sarason is being investigated. Sub-symbols on operators densely defined. Analytical closures are shown. Toeplitz is thickly defined.

Key words:

Toeplitz Operator, Hardy Space, sarason sub-spaces, Analytical Functions.

1.Introduction

A limited Toeplitz operator has been studied over the Hardy area, where equivalent Toeplitz operato exist. We define a Toeplitz limited operator to extend the Toeplitz matrix definition. If r definitions the matrix is representation of the operator, we call T is Toeplitz operator, with respect to the orthonormal basis $\{e^{in\theta}\}_{n=0}^{\infty}$ is constant along the diagonals. Algebraically, we denoted by $T = S^{*}TS$,the shift operator for the Hardy space is denoted by $S = M_z$ .We used a coefficients corresponding to each diagonal of the matrix are the Fourier coeficients of a function, hence $\varphi \in L^{\infty}(T) \Rightarrow P_{H^2(T)} M_{\varphi}$. Let $P_{H^2(T)} : L^2(T) \to H^2(T)$ is the projection, and $M_{\varphi} : H^2(T) \to L^2(T)$ is the bounded multiplication operator given by:

$$\sum_m M_{\varphi} f^m = \varphi \sum_m f^m \quad (1)$$

Finally, the exact opposite is true, if the bounded operator given by:

$T_{\varphi} = P_{H^2(T)} M_{\varphi}, \varphi \in L^{\infty}(T) \Rightarrow T = S^{\bullet} T S$ (2)

The corresponding definitions of Toeplitz operators are no longer equivalent if the condition is closed and densely defined. For example, if the coefficients of an upper triangular matrix are the coefficients of a Smirnov class function $\varphi \in N^{+}$, Then a densely defined operator defined by the closure of the matrix, call it T ,and the operator will be a deputy of the densely defined (Toeplitz analytic) multiplication operator M_{φ}. In contrast to its limiting counterpart, the Multiplication Operator T cannot be represented with $PM_{\bar{\varphi}}$, since the domain is strictly greater than the domain of $M_{\bar{\varphi}}$. The following algebraic equations are encountered by operator T :

(i) $D^m(T)$ is S -invariant,

(ii) $T = S^{\bullet} T S$,

(iii) If $f^m \in D^m(T), \sum_m f^m(0) = 0 \Rightarrow \sum_m S^{\bullet} f^m \in D^m(T)$

The densely defined analog of the algebraic condition of the limited Toeplitz operators can be seen. Consequently, T fulfills the Algebraic requirements for Toeplitz but is not a multiplication Toeplitz operator. T is a closed extension of the Toeplitz type multiplication operator, however, the following question was raised at the end of [1].

Question 1: Is it possible to characterize those closed densely defined operators T on $H^2(T)$ with the above three properties? Moreover, is every closed densely defined operator on $H^2(T)$ that satisfies these conditions determined in some sense by a symbol? we aims to address the second half of this question. If a closed densely defined operator, T, satisfies the three algebraic conditions above, henceforth a Sarason-Toeplitz operator, then is T the extension of an operator of the form PM_{ϕ} where M_{ϕ} is a densely defined multiplication operator $M_{\phi} : H^2 \to L^2$?For bounded Toeplitz operators the recovery of the symbol of a Toeplitz operator can be achieved through the symbol map on $\mathcal{T}$, the algebra of generated by the collection of Toeplitz operators in $\mathcal{L}(H^2)$. Douglas demonstrated that there is a unique multiplicative mapping $\phi : \tau \to L^{\infty}$ such that $\phi \sum_m (T_{f^m} T_{g^m}) = \phi(\sum_m T_{f^m})(\sum_m T_{g^m}) = f^m g^m$ in [2, 3]. This fact was proven again in [4] by Halmos and Barria using the limits along the diagonals of a Toeplitz matrix in order to find the symbol in L^{∞}. The Hardy space can be identified with analytic functions of the disc $\mathbb{D}$ such that the Taylor coefficients of these functions are square summable. By this view point, H^2 is a reproducing kernel Hilbert space (RKHS) over $\mathbb{D}$ with the kernel functions $k_w(z) = (1 - \bar{w}z)^{-1}$ for $|w| < 1$. In the case of bounded Toeplitz operators, the Berezin transform, a tool particular to the study of RKHSs, is sufficient for the recovery of the of L^{∞} functions via radial limits of the Berezin transform of a bounded Toeplitz operator [5]. However, in more general cases the recovery of the symbol of a Sarason-Toeplitz operator is no longer clear. The recovery of the symbol of a densely defined analytic (or a co-analytic) Toeplitz operator with symbol ϕ can be accomplished by the use of the Berezin transform. In this case, the adjoint of an analytic Toeplitz operator has the reproducing kernels as eigenvectors, k_z, with eigenvalues $\overline{\phi(z)}$ [1]. Thus $\tilde{T}(z) = (1 - |z|^2)\langle k_z, T^* k_z\rangle = (1 - |z|^2)\,\langle k_z, \overline{\phi(z)} k_z\rangle = \phi(z)$. The application of the Berezin transform requires the kernel functions $k_w(z) = (1 - \bar{w}z)^{-1}$ to be in the domain of operator or in the domain of its adjoint. Thus, the investigation of a new method is justified for the recovery of the symbol of a densely defined Sarason-Toeplitz operator .We introduce the Sarason Sub-Symbol, which depends on a choice of a function in $D(T)$, as a family of symbol maps for Sarason-Toeplitz operators. In the development, it will be demonstrated that for the bounded case the Sarason Sub-Symbol is unique iff the operator is Toeplitz. Thus the uniqueness of the Sarason Sub-Symbol provides another equivalent definition for a bounded Toeplitz operator. Subsequently it is demonstrated that the Sarason Sub-Symbol for an analytic

Toeplitz operator is unique and determines the operator. The rest we concerned with classes of Toeplitz operators for which the existence of the Sarason Sub-Symbol can be established, and it demonstrates sufficient conditions to show that T is a closed extension of a multiplication type Toeplitz operator.

2. Sarason problem

The Sarason problem has been solved for long time for bounded Toeplitz operators in [6, 2].Indeed, if a Toeplitz operator is bounded, then it can be represented by an L^{∞} function. . All densely defined closed operators on $H^2(T)$ which travel with the shift operator deputy [7], have been characterized by Suarez, and Sarason gives the operators traveling with the shift operator so-
called analytical Toeplitz operators a different treatment. These two operator's collections meet the conditions of Sarason-Toeplitz. In addition, the analytic Toeplitz operators are precisely the operators of multiplication by an function in the Smirnov class, N^+ [1].
The operators of Suarez are the deputy members of these Toeplitz analytical operators, and are refer red to as Toeplitz coanalytic operators [7]. Thus the above classes of Sarason-Toeplitz operators are completely characterized by a symbol.
Analytically and jointly analytically The operators of Toeplitz both meet the requirements of the Sar ason.
This relationship is generalized by the following.

Proposition (2.1): If T is a Sarason-Toeplitz operator then T^* is also Sarason-Toeplitz operator.
Proof: T is a closed operator that is densely defined, which means T^* is also closed. $D^m(T^*)$ therefore, is not empty. We proved T^* has a shift invariant domain by using $g^m \in D^m(T^*)$ .this definition this means that $\sum_m \tilde{L}(f^m) = \sum_m \langle Tf^m, g^m \rangle$ is a continuous functional. Let $zg^m \in D^m(T^*)$ to prove $\sum_m Lf^m = \sum_m \langle Tf^m, zg^m \rangle$ is continuous. Since $zD^m(T) \subset D^m(T)$, and $zD^m(T)$ has co-dimension 1 in $D^m(T)$. Thus there exists $f_0^m \in D^m(T)$ such that $D^m(T) = c\{f_0^m\} \oplus zD^m(T)$ .The functional L is continuous on $C\{f_0^m\}$, since it is finite dimensional. Therefore, it suffices to show that L is continuous on $zD^m(T)$. If $f^m = zh^m, h^m \in D^m(T)$, then
$\sum_m Lf^m = L\sum_m zh^m = \sum_m \langle Tzh^m, zg^m \rangle = \sum_m \langle Th^m, g^m \rangle = \tilde{L} \sum_m h^m$ .Thus L is continuous on $zD^m(T)$, since $\tilde{L}$ is continuous on $D^m(T)$. Now suppose that $g^m \in D^m(T^*)$ and $\sum_m g^m(0) = 0$,and consider the functional $\sum_m L_2(f^m) = \sum_m \langle Tf^m, S^{\bullet} g^m \rangle$ defined for $f^m \in D^m(T)$. This functional can be rewritten as
$\sum_m L_2(f^m) = \sum_m \langle S^{\bullet} TSf^m, S^{\bullet} g^m \rangle = \sum_m \langle TSf^m, g^m \rangle := \tilde{L}_2 \sum_m Sf^m$ .It follows that $L_2(f^m)$ is continuous, since $\tilde{L}_2(Sf^m)$ is continuous with respect to f^m. Finally for all $f^m \in D^m(T^{\bullet}),\ g^m \in D^m(T)$ we have
$\sum_m \langle T^{\bullet} f^m, g^m \rangle = \sum_m \langle f^m, Tg^m \rangle = \sum_m \langle f^m, S^{\bullet} TSg^m \rangle = \sum_m \langle S^{\bullet} T^{\bullet} Sf^m, g^m \rangle$, which results in the second state.

3.The Sarason Sub-Symbol

While it is possible to retrieve the symbol of densely defined Toeplitz analytical and co-analytical operators by using the Berezin transform, it is not obvious whether to recover the symbol of more general and densely operators defined for Toeplitz. Because the functions k_z are required for Berezin to be well defined in either the operator's domain or the operator's deputy. The sub-symbol of Sarason is introduced instead as a candidate to retrieve the Sarason-Toeplitz densely defined operators' symbol. For the Sarason Sub-Symbol, as a motivating example, first assume that T is a bound Toeplitz operator with after all $\varphi \in L^{\infty}$. In this case $a_n = \begin{cases} \langle T1, Z^n \rangle & ,n \geq 0 \\ \langle TZ^n, 1 \rangle & ,n < 0 \end{cases}$ are the Fourier coefficients of φ. Thus φ can be reconstructed as follows $\varphi(e^{i\theta}) = \sum_{n=1}^{\infty} \langle Tf^m z^n, 1 \rangle e^{-in\theta} + \sum_{n=0}^{\infty} \langle T1, z^n \rangle e^{in\theta}$. While it is not expected that $1 \in D^m(T)$ in general, given any function $f^m \in D^m(T)$ the domain of the densely defined operator TM_{f^m} contains the polynomials, since $D^m(T)$ is shift invariant. The Sarason Sub-Symbol is defined as follows:

Definition (3.1): Let T be an operator with a shift invariant domain $D^m(T)$. For $f^m \in D^m(T) \setminus \{0\}$ the Sarason Sub-Symbol corresponding to f^m is given by

$\sum_m R_{f^m} = \sum_m h_{f^m} \setminus f^m, \sum_m h_{f^m} = \sum_{m=0}^{\infty} \langle Tf^m Z^n, 1 \rangle e^{-in\theta} + \sum_{m=0}^{\infty} \langle Tf^m, Z^n \rangle e^{in\theta}$, Where in a certain sense this series is convergent. Sub-symbol f^m of the partial Sarason is given by

$\sum_m R_{f^m,N} = \sum_m h_{f^m,N} \setminus f^m, \sum_m (\sum_{n=1}^{N} \langle Tf^m Z^n, 1 \rangle e^{-in\theta}) + \sum_m (\sum_{n=0}^{\infty} \langle Tf^m, Z^n \rangle e^{in\theta})$.

Heuristically, if T is a Toeplitz operator associated with multiplication by the symbol ϕ, then $\sum_m h_{f^m} = \varphi . \sum_m f^m$. The issue of the Sarason sub-well symbol's definedness depends on the convergence of the series in the definition of h_{f^m}. When $\sum_m h_{f^m} = \varphi . \sum_m f^m$ $, \varphi \in L^{\infty}$ and is a well-defined function in L^2. In particular, with the Sarason Sub symbol we can characterize all Toeplitz bounded operators.

Proposition (3.2): Let V be a bounded operator on H^2. The operator V is a Toeplitz operator if f the Sarason Sub-Symbol is independent of the choice of $f \in H^2$.

Proof: Suppose that $V = T_{\varphi}$ is a Toeplitz operator with symbol φ.

$$\sum_m h_{f^m} = \sum_m \sum_{n=1}^{\infty} \langle T_{\varphi} f^m Z^n, 1 \rangle_{H^2} e^{-in\theta} + \sum_m \sum_{n=1}^{\infty} \langle T_{\varphi} f^m, Z^n \rangle_{H^2} e^{-in\theta} + \sum_m \sum_{n=0}^{\infty} \langle T_{\varphi} f^m, Z^n \rangle_{L^2} e^{-in\theta} = \varphi \sum_m f^m$$

Thus $\sum_m R_{f^m} = \sum_m h_{f^m} \setminus f^m = \varphi$ is independent of the choice of f^m. Let V is not an operator of Toeplitz. That means that there are a couple of integers $n, m \in N, n < m$ (without loss of generality) and $\langle VZ^n, Z^m \rangle \neq \langle V1, Z^{m-n} \rangle$. In this case consider the two Sarason Sub-Symbols

$$R_1 = \sum_{k=1}^{\infty} \langle VZ^k, 1 \rangle e^{-ik\theta} + \sum_{k=0}^{\infty} \langle V1, Z^k \rangle e^{ik\theta} = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, R_{Z^n} = e^{-in} (\sum_{k=1}^{\infty} \langle VZ^{n+k}, 1 \rangle e^{-ik\theta} + \sum_{k=0}^{\infty} \langle VZ^n, Z^k \rangle e^{ik\theta}) = e^{-in\theta} \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}$$

The difference of the two sub-symbols yields $R_1 - R_{Z^n} = e^{-in\theta}(\sum_{k=-\infty}^{\infty}(a_k - b_k)e^{ik\theta})$

The coeficient $(a_{m-n} - b_m) \neq 0$ by construction. Therefore $R_1 \neq R_{Z^n}$. So the uniqueness of its Sarason Sub-Symbols characterizes every bounded Toeplitz operator. This motivates the study of densely defined operators. The following are interactions between Sarason sub-symbols and Sarason-Toeplitz operators, which are thoroughly defined.

4. Densely Analytical on Toeplitz Operators

Just like in Proposition (3.2), a symbol completely characterizes the analytical densely defined Toeplitz operator. As indicated in [1], these operators are precisely symbolic φ multiplication operators, in the Smirnov functional class. In other words, the ratio of H^{∞} ,to functions $\frac{b}{a}$ can be written for every $|a(e^{i\theta})|^2 + |b(e^{i\theta})|^2 = 1$ for each φ| and an external function. The Sarason sub-symbol is unique in this configuration.

Theorem (4.1): Given a Sarason-Toeplitz operator T, there exists a symbol $\varphi \in N^+$ for which $T = M_{\varphi} \Leftrightarrow \langle TZf^m, 1\rangle = 0, \forall f^m \in D^m(T)$, in addition, the sub symbol for Sarason is singular.

Proof: The forward direction follows since $T_{\varphi} = M_{\varphi}, \varphi \in N^+$. This means $TS = ST$, and $\langle TZf^m, 1\rangle = \langle ZTf^m, 1\rangle = 0, 1 \in (zD^m(T))^{\perp}$?. In order to establish sufficiency, let $f_1^m = \sum_m \sum_{n=0}^{\infty} a_n z^n, f_1^m = \sum_m \sum_{n=0}^{\infty} b_n z^n \in D^m(T) \setminus \{0\}$. By hypothesis, $\sum_m h_{f_i^m} = \sum_m \sum_{n=0}^{\infty} \langle Tf_i^m 1, Z^n\rangle Z^n = T\sum_m f_i^m \in H^2, i = 1, 2.$ In order to establish uniqueness of the symbol, $R_{f_1^m} = R_{f_2^m}$, consider the function $h_1 f_2^m = h_2 f_1^m \in L^1(T)$. The Fourier series of $h_1 f_2^m, h_2 f_1^m$, can be computed through convolution. Hence

$h_1 \sum_m f_2^m = \sum_m (\sum_{n=0}^{\infty}(\sum_{k=0}^{\infty}\langle Tf_1^m, Z^{n-k}\rangle b_k))Z^n = \sum_m (\sum_{n=0}^{\infty}(\sum_{k=0}^{n}\langle TZ^k f_1^m, Z^n\rangle b_k))Z^n$, and

$S^{\bullet}TSf^m = Tf^m, h_2 \sum_m f_1^m = \sum_m \sum_{n=0}^{\infty}(\sum_{k=0}^{n}\langle TZ^k f_2^m, Z^n\rangle a_k)Z^n)$ .The second equality follows since, and $\langle TSf^m, 1\rangle = 0 \Rightarrow TSf^m(0) = 0, SS^{\bullet}TSf^m = TSf^m$. This leads to

$H := h_1 \sum_m f_2^m - h_2 \sum_m f_1^m = \sum_m (\sum_{n=0}^{\infty}(\sum_{k=0}^{n}\langle TZ^k f_1^m, Z^n\rangle b_k - \sum_{k=0}^{n}\langle TZ^k f_2^m, Z^n\rangle a_k)Z^n))$. In order to establish that each coefficient is in fact zero, consider, for arbitrary n, the coefficient of $Z^n : \tilde{H}(n) = \sum_m (\sum_{k=0}^{n}\langle TZ^k f_1^m, Z^n\rangle b_k - \sum_{k=0}^{n}\langle TZ^k f_2^m, Z^n\rangle a_k) = \sum_m T \left\langle (f_1^m[\sum_{k=0}^{n} b_k Z^k - f_2^m] - f_2^m[\sum_{k=0}^{n} a_k Z^k - f_2^m]), Z^n \right\rangle$

The H^2 function inside of T is in fact in the domain of T by the properties of Sarason-Toeplitz operators, and this function has a zero of order greater than n at zero. Denote by $Z^{n+1}F_n^m$, the function in the argument of T . By our hypothesis,

$\tilde{H}(n) = \sum_m \langle TZ^{n+1}F_n^m, Z^n\rangle = \sum_m \langle TZF_n^m, 1\rangle = 0$ .Therefore $R_{f_1^m} = R_{f_2^m}$ for any choice of $f_1^m, f_2^m \in D^m(T) \setminus \{0\}$,so let $\varphi \in R_{f_1^m}$ be the proposed symbol for the Sarason-Toeplitz operator

$T.h_{f^m} = T_{f^m} \in H^2, \forall f^m \in D^m(T)$.Further, given any $z \in D^m\ \exists f_z^m \in D^m(T): f^m(z) \neq 0$ (this follows from the density of $D^m(T) \in H^2$). Thus $\varphi = \sum_m (Tf_z^m \setminus f_z^m)$ is analytic at z for every point $z \in D^m$. Finally note that for each $f^m \in D^m(T), M_\varphi \sum_m f^m = \varphi.\sum_m f^m = \sum_m (Tf^m \setminus f^m) f^m = T\sum_m f^m\ T$. Thus $T = M_\varphi$ is a densely defined multiplication operator with an analytic symbol. By [1], $\varphi \in N^+$.

Corollary (4.2): A Sarason-Toeplitz operator T on is analytic $(ST = TS) \Leftrightarrow \langle T_z f^m, 1\rangle \forall f^m \in D^m(T)$.

5.Symbols that are Ratios of L^2 Functions and H^2 Functions

When the Toeplitz operator with a symbol φ is analytically densely defined (expressed as $\varphi = b/a$ in canonical form), the domain is given by $D^m(T) = aH^2$. This means that there is an outer function, in particular a, in the domain of T. Moreover, since $T = M_\varphi$, it is clear that $h_a = \varphi.\, a \in H^2$. Therefore, the existence of an outer function $f^m \in D^m(T)$ for which h_{f^m} is well defined is straighforward in the case of analytic Toeptliz operators. When we look at a Toeptliz coanalytic form operator $M_\varphi^\bullet$, its domain is given by $\mathrm{H}(b)$,the de Branges-Rovnyak space corresponding to $b.\mathrm{H}(b)$ contains the space $a.H^2$ as a subspace. Hence, in its domain, it also has an external function. Especially, if $f^m = a.p$, where p is a polynomial, then h_{f^m} (corresponding to $M_\varphi^\bullet$) is in L^2. Since a function is external, the collection f^m is very compact in H^2. Therefore, the set $D_2^m(T) = \{f^m \in D^m(T): h_{f^m} \in L^2\}$ is dense in $D^m(T) = D^m(M_\varphi^\bullet) = \mathrm{H}(b)$. For general Sarason-Toeplitz operators, $D_2^m(T)$ the answer to the problem of nonemptiness (as well as density) is unknown. The Sarason Sub symbol's applicability is extended to include form functions B/A where $B \in L^2$ and A is an H^2 outer function.

Lemma (5.1): Let φ be a function on the unit circle that can be written as the ratio of an L^2 function and an H^2 outer function. Let $D^m(M_\varphi) = \{f^m \in H^2 : \varphi.f^m \in L^2\}$ .The operator $M_\varphi : D^m(M_\varphi) \to L^2$ is a closed densely defined operator on H^2.

Proof: Write $\varphi = B/A$,where $B \in L^2, A \in H^2$ is an outer function. Since $B.p \in L^2$ for every polynomial $p(z)$, we see that $A.p \in D^m(M_\varphi)$ for every polynomial p. Therefore, $D^m(M_\varphi)$ is dense in H^2 by the outer property of A. Now suppose that $\{f_n^m\} \subset D^m(M_\varphi), f_n^m \to f^m \in H^2$,Suppose further that $M_\varphi f_n^m \to F^m \in L^2$. Since $f_n^m \to f^m$ in the L^2 norm, there exists a subsequence, $\{f_{n_j}^m\}$, such that $f_{n_j}^m \to f^m$ almost everywhere. Since A is an outer function, $A(e^{i\theta}) \neq 0$ for almost every θ. Thus $\varphi f_{n_j}^m \to \varphi f^m$ almost everywhere. The subsequence $\varphi f_{n_j}^m \to F^m$ in L^2 and so there is a subsequence $\varphi f_{n_{j_k}}^m \to F^m$ all most everywhere. Since, this subsequence is converges to φf^m almost everywhere . Thus we may conclude that $\varphi f^{m} = F^{m}$ almost everywhere, which completes the proof.

Theorem (5.2): Let T be a Sarason-Toeplitz operator. If there is an H^{2} outer function $f^{m} \in D^{m}(T)$ such that $\sum_{m}(\sum_{n=0}^{\infty}\langle Tf^{m}Z^{n},1\rangle Z^{-n}) \in L^{2}$, then T extends a closed densely defined operator of the form $T_{\varphi} = PM_{\varphi}$ where $\varphi = R_{f^{m}}$ is the ratio of an function and an H^{2} outer function. Moreover, $D_{2}^{m}(T)$ is a dense subset of $D^{m}(T)$.

Proof: Let f^{m} be an H^{2} outer function in $D^{m}(T)$, and let $h_{f^{m}}$ be the corresponding numerator of the Sarason Sub-symbol corresponding to f^{m}. Express $h_{f^{m}} = \sum_{n=-\infty}^{\infty} b_{n}Z^{n}$. By the properties of Sarason-Toeplitz operators, $b_{n-m} = \sum_{m}\langle Tf^{m}Z^{m}, Z^{n}\rangle$. Now consider the operator $TR_{f^{m}} = PM_{R_{f^{m}}}$,which is closed and densely defined by Lemma (5.1). Since the domain of T is shift invariant, $f^{m}.P \in D^{m}(T)$ for every polynomial p. Moreover, $h_{f^{m}}.p \in L^{2}$ for every polynomial p. It follows that $D_{2}^{m}(T) \subset D^{m}(T)$ is dense in H^{2} since f^{m} is an outer function. Define the set $F^{m} := \{f^{m}.P\} \subset D_{2}^{m}(T)$ Let $p(z) = a_{0} + a_{1}z + ... + a_{k}z^{k}$ be a polynomial of degree $k \in N$. The product of $h(z), p(z)$ can be calculated as follows:

$$h(z).p(z) = \sum_{n=-\infty}^{\infty}(\sum_{m=0}^{k} b_{n-m}.a_{m})Z^{n} = \sum_{n=-\infty}^{\infty}(\sum_{m=0}^{k}\langle T\, a_{m}Z^{m}, Z^{n}\rangle)Z^{n} = \sum_{m}(\sum_{n=-\infty}^{\infty}\langle Tf^{m}p, Z^{n}\rangle)Z^{n} = \omega(z) + T(f^{m}p)(z)$$

Where $\omega(z) = \bar{H}_{0}^{2}, h_{f^{m}} \in L^{2}$. In particular, this means $T_{R_{f^{m}}}(f_{p}^{m}) = p(h_{p}^{m}) = T(f_{p}^{m})$ for all polynomials p. Hence, T agrees with T_{R_f} on a dense domain, and T extends $T_{R_f}\big|_{F}$. Finally, by Lemma (5.1), $T_{R_f}\big|_{F}$ is closable, and $T_{R_{f^{m}}}\big|_{F^{m}} \subset T \Rightarrow T_{R_{f^{m}}}\big|^{\bullet}_{F^{m}} \subset T^{\bullet\bullet} = T$. The above theorem relies on the ability to find an outer function in $D_{2}^{m}(T)$. Once such a function is found, T The respective operator $M_{R_{f^{m}}}$ is shown to be a closed extension. When such a function does not exist, it can be shown that T is the Toeplitz type multiplication limit on a limited domain.

Proposition (5.3): Suppose T is a Sarason-Toeplitz operator, let $f^{m} \in D^{m}(T)$, and define $F^{m} = \{f^{m}.p\}$, p is polynomial. There exists a sequence of multiplication type Toeplitz operators, $T_{\varphi M} = PM_{\varphi M}$ such that $T_{\varphi M}$ is strongly converges to T on all of F^{m}. In addition, they have a common dense domain.

Proof: Let $f^{m} \in D^{m}(T)$ and let $p(z) = a_{0} + a_{1}z + ... + a_{k}z^{k}$, $h_{f^{m},N}(z)p(z)$ be a polynomial of degree k . Now, as in Theorem (5.2), consider the product

$$h_{f^{m},N}(z), p(z) : h_{f^{m},N}(z).p(z) = \sum_{n=-N}^{\infty}(\sum_{m=0}^{\min(K,n+N)} b_{n-m}a_{m})Z^{n} = \sum_{m}(\sum_{n=-N}^{K-N-1}\langle Tf^{m}(a_{0} + a_{1}z + ... + a_{n} + NZ^{n} + N, Z^{n})\rangle Z^{n}) + \sum_{m}(\sum_{n=K-N}^{\infty}\langle Tf^{m}p, Z^{n}\rangle Z^{n})$$

Therefore,

$$\sum_{m}T_{R_{f^{m},N}}(f^{m}p)(z) = p(h_{N}P)(z) = \sum_{m}(\sum_{n=-N}^{K-N-1}\langle Tf^{m}(a_{0} + a_{1}z + ... + a_{n} + NZ^{n} + N, Z^{n})\rangle Z^{n}) + \sum_{m}(\sum_{n=\min(K-N,0)}^{\infty}\langle Tf^{m}p, Z^{n}\rangle Z^{n})$$

The left sum is empty for large enough N, therefore $\left\{T_{R_{N,f^m}}(f^m p)\right\}$ is constant for large enough N. This means $T_{R_{N,f^m}}(f^m p) \to T(f^m p), N \to \infty$. In order for each Toeplitz operator to find a common domain, consider the inner-outer factorization $f^m = f_i^m f_0^m$. The functions, $\tilde{h}_N = h_{f^m,N} / f_i^m \in L^2$ since f_i^m has modulus 1 on the circle. Thus $\tilde{h}_N f_0^m p \in L^2$ for all polynomials p. $\Rightarrow F_0^m = \left\{f_0^m p\right\} \subset D^m(T_{R_{N,f^m}}) \forall N$.

6. Example of a Non-Sarason Toeplitz Operator

This shows that a densely defined matrix of Toeplitz does not necessarily identify a Sarason-Toeplitz operator. We especially extend a high-triangle Toeplitz matrix and show that the extension domain does not invariantly change. An upper triangular Toeplitz matrix is a matrix of the form $\begin{pmatrix} Y_0 & Y_1 & Y_2 \\ 0 & Y_0 & Y_1 \\ 0 & 0 & Y_0 \end{pmatrix}$ This matrix has a natural dense domain, namely polynomials, as an operator over H^2. The domain density does not depend on the sequence. Following Sarason [1], this operator may be extended as $T f^m = \sum_{m=0}^{\infty}(\sum_{n=0}^{\infty} Y_n \tilde{f}^m(n+m) Z^n)$,where the domain of T is the collection of functions in H^2 for which $T f^m \in H^2$.

Lemma (6.1): The sequence $\left\{c_m = \sum_{n=1}^{\infty}(n+1)^{-m}\right\}_{m=2}^{\infty} \in l^2$.

Proof: Each term of the sequence can be bounded by $\int_0^{\infty}(x+1)^{-m}\,dx = \dfrac{1}{m-1}$.Thus, c_m is bounded by an l^2 sequence, and so it is also l^2.

Theorem (6.2): Let T be the extension of an upper triangular matrix given by

$$Tf^m = \sum_m (\sum_{m=0}^{\infty} n!\tilde{f}^m(n+m) Z^n)$$

The domain of T is defined to be $D^m(T) = \left\{f^m \in H^2 : Tf^m \in H^2\right\}$. Every function $f^m \in D^m(T)$ is an entire function and P can be written as $f^m(z) = \sum_m (\sum_{n=0}^{\infty} a_n \dfrac{Z^n}{n!})$,where $\sum_{n=0}^{\infty} a_n$ converges.

Proof : First suppose that $f^m \in D^m(T)$. By definition, the zeroth coefficient of Tf^m is given by $n!\tilde{f}^m(n)$, which must be a convergent series. Declaring $a_n = n!\tilde{f}^m$, it can be seen that $\sum a_n$ converges. Moreover, since $f^m(z) = \sum_m (\sum_{n=0}^{\infty} a_n \dfrac{Z^n}{n!})$, the function f^m must be an entire function. Define $d_0 = \sum_m (\sum_{n=0}^{\infty} n!\tilde{f}^m(n)) = \sum_{n=0}^{\infty} a_n$. Note that since a_n converges so does

$\sum_{n=0}^{\infty} a_n b_n$ for any positive monotonically decreasing sequence $\{b_n\}$. Thus for each $m = 1,2,3,...$ the series $d_m = \sum_m (\sum_{m=0}^{\infty} n!\tilde{f}^m (n+m)) = \sum_{n=0}^{\infty} \frac{a_{n+m}}{(n+1)(n+2).....(n+m)}$ converges. This enables us to define Tf^m formally as $\sum_{m=0}^{\infty} d_m Z^m$. In order to demonstrate that dm is in (l3), write dm as follows: $d_m = \frac{a_m}{n!} + \sum_{n=1}^{\infty} \frac{a_{n+m}}{(n+1)(n+2).....(n+m)} := S_m + t_m$ .The sequence $\{S_m\} \in l^2$,since $a_m \to 0$. The sequence $\{t_m\}$ is bounded by $c_m = \sum_{n=0}^{\infty} (1+n)^m$ for sufficiently large *m*, since $|a_m| < 1$ for sufficiently large. By Theorem (6.2), t_m is in l^2. This completes the proof of the theorem.

Corollary (6.3): The operator domain in Theorem (6.2) is not invariant with transformations.

Proof: The function $f^m(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} . \frac{z^n}{n!} \in D^m(T)$, since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, converges. Now consider the function $zf^m(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} . \frac{z^n + 1}{n!} = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} . \frac{z^n}{(n-1)!} = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n-1} . \frac{z^n}{n!}$ .The series $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{n}{n-1}$ does not converge, which means $zf^m(z) \notin D^m(T)$. By applying the same techniques used in proving Theorem (6.2). If there is a slightly weaker result, $n!$ is replaced by a sequence of complex numbers $\{Y_n\}$ with the growth condition $|Y_{n+1}| > (n+1)|Y_n|$.

Theorem (6.4): Let $\{Y_n\}$, be a complex number sequence as described above, and define the operator $f^m(z) = D^m(T) = \{f^m \in H^2 : Tf^m \in H^2\}$ with the domain . The operator T is densely defined and functions of the form $f^m(z) = D^m(T) = \{f^m \in H^2 : Tf^m \in H^2\}$,

$f^m(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{Y_n}, \sum |a_n| \to \infty$

are in its domain.

7. Conclusion

we aim to give a partial answer to a question posed by Donald Sarason in [1]. Sarason asked if a densely defined closed operator which meets certain algebraic characteristics similar to a limited Toeplitz operator is defined in some way by a symbol. We call these operators Sarason-Toeplitz operators. The Sarason sub-symbol was presented as a family of potential symbol maps for Sarason-Toeplitz operators. For bounded and analytic densely defined Toeplitz operators, the Sarason sub-symbol is unique and characterizes the operators. For a coanalytic densely defined Toeplitz operator, the Sarason sub-symbol produces a densely-defined Toeplitz operator that agrees to a restricted domain with the original Toeplitz coanalytic operator. Finally, these results were extended to a broader class of Sarason-Toeplitz operators, provided their domains contain functions that are ratios of L^2 functions and H^2 outer funtions. We focused on a densely defined coanalytic Toeplitz matrix. The area of the matrix extension had been fully classified, and the densely defined operator was shown not to be of the Sarason-Toeplitz type. This demonstrates that the definition of a Toeplitz matrix and that of a Sarason-Toeplitz operator do not coincide The density of set $D_2^m(T)$ is unknown to general Sarason-Toeplitz operators T. We also do not know if there are nontrivial elements in this set. This is a question for future research and probably for future research, if it is to be proven true, the closedness of T must be leveraged.

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